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Modulation of Entanglement for Coupled Superconducting Qubits Under Non-Markovian Environment

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Abstract The evolution of entanglement decoherence is investigated for a coupled superconducting qubit under non-Markovian environment by utilizing a commensal entanglement degree. The results show that, owing to the memory feedback effect of environment, the entanglement degree of the coupled qubits at the thermal equilibrium always monotonously tends to zero so that entanglement sudden death occurs briefly in the non-Markovian process. Different from the Markovian process, stronger the dissipation is, faster the entanglement sudden death is. We find that, furthermore, the interaction between the qubits results generally in reduction of entanglement degree in the quantum system. With some special initial states or initial phase angles, however, the influence of the interaction between qubits on the system entanglement degree can be avoided.

Keywords Entanglement · Coupled qubits · Non-Markovian process · Concurrence

1 Introduction

As a physical resource, entangled state plays an important role in many aspects of quantum information such as quantum teleportation, quantum key distribution and quantum computation [1–5]. However, the quantum characteristics of the physical system, depended on the experimental conditions and influence from unavoidable experimental noise, are fragile so

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as to all prepared entangled states not to maximum entangled state. Otherwise, pure entangled state will degenerate to mixed state in real physical world because of environmental decoherent effect. Quantum communication and quantum computation with such mixed entangled state lead to the information distortion [6–10]. It was found that, indeed, a mixed state of an initially entangled two qubit system, under the influence of a pure dissipative environment, becomes completely disentangled in a finite time. This was called as entanglement sudden death (ESD) and was recently observed in two elegantly designed experiments with photonic qubits and atomic ensemble. In order to ensure the realization of the correct computing of quantum logic and quantum computation, people have made a lot of theoretical and experimental study for the phenomenon of ESD and shown how to avoid ESD and produce the scheme of sudden birth for some cases [10–13].

Even though numerous investigations on ESD in a variety of systems have been done so far, the question of ESD in interacting qubits remains yet open, especially for the superconducting quantum bit system which is very sensitive for the environment [14–18]. Due to memory effect, the effect on the dynamics evolution of quantum system made by the environment is closely related to evolutionary history of the system. Thus, it is interesting to in-depth study the non-Markov dynamic evolution of the superconducting qubit system. Based on spin-boson model, in this paper, we investigate this question of ESD for a system of interacting superconducting flux qubits coupling with the non-Markov environment.

2 Non-Markov Master Equation

Consider a pair of initially entangled interacting superconducting flux qubits, where the flux qubits interacts independently with the thermal environment. In the interaction picture, the Hamiltonian for our model is then given by ($k_B = \hbar = 1$)

$$H_I(t) \equiv H_I^{(s)}(t) + H_I^{(d)}(t), \tag{1}$$

where

$$H_I^{(s)}(t) = k(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+), \tag{2}$$

$$H_I^{(d)}(t) = (\sigma_1^+ + \sigma_2^+) \sum_n g_n b_n e^{i(\varepsilon - \omega_n)t} + (\sigma_1^- + \sigma_2^-) \sum_n g_n^* b_n^+ e^{-i(\varepsilon - \omega_n)t}, \tag{3}$$

where k is a strength of the qubit–qubit interaction and$g_n$are coupling constants of the interaction between qubit and the local reservoir. While the Hamiltonian of the bosonic bath is characterized by the annihilation and creation operators b_n and b_n^+ and $\sigma_j^{\pm}$ ($j = 1, 2$) are the Pauli operators with the angular momentum commutation algebra.

At first, we investigate the dynamics of entanglement of the qubits in a thermal reservoir. The time evolution of the density operator ρ, which gives us an information about the dynamics of the system, may be evaluated from the integro-differential quantum-Liouville equation of motion

$$\frac{d}{dt}\rho(t) = -i[H_I^{(s)}(t), \rho(0)] - \int_0^t d\tau Tr_B([H_I(t), [H_I(\tau), \rho(t) \otimes \rho_B]]). \tag{4}$$

Without loss of generality, the capacity of the heat storage R is assumed to be very large so that, during the process of dynamic evolution, quantum system has little effect on the heat storage. Thus the heat storage has been in its own thermal equilibrium state all the time, i.e.,

$$\rho_T(t) = \rho(t) \otimes \rho_R. \tag{5}$$

At the initial time $t = 0$, in addition, the heat storage only has diagonal entry in the photon number representation and the $H_I(t)$ will absorb or release a photon, which lead to

$$Tr_B([H_I^d(t), \rho(0) \otimes \rho_B]) = 0,$$

in terms of (4) and (5), the involving equation of dynamics for density operator may be rewritten as

$$\begin{aligned}\frac{d}{dt}\rho(t) &= -i[H_I^{(s)}(t), \rho(0)] - t[H_I^{(s)}(t), [H_I^{(s)}(t), \rho(t)]] \\ &\quad + B(t)\{[\sigma_1^- \rho(t), \sigma_1^+] + [\sigma_2^- \rho(t), \sigma_2^+]\} + B^*(t)\{[\sigma_1^-, \rho(t)\sigma_1^+] + [\sigma_2^-, \rho(t)\sigma_2^+]\} \\ &\quad + A(t)\{[\sigma_1^+, \rho(t)\sigma_1^-] + [\sigma_2^+, \rho(t)\sigma_2^-]\} + A^*(t)\{[\sigma_1^+ \rho(t), \sigma_1^-] + [\sigma_2^+ \rho(t), \sigma_2^-]\},\end{aligned} \tag{6}$$

where,

$$A(t) = i\sum_n |g_n|^2 n(\omega_n)\frac{1 - e^{i(\varepsilon-\omega_n)t}}{\varepsilon - \omega_n}, \qquad B(t) = i\sum_n |g_n|^2[n(\omega_n) + 1]\frac{1 - e^{i(\varepsilon-\omega_n)t}}{\varepsilon - \omega_n}.$$

In order to simplify the calculation, we only consider the $T = 0$ K thermal state with the correlation functions,

$$A(t) = 0, \qquad B(t) = i\sum_n |g_n|^2 \frac{1 - e^{i(\varepsilon-\omega_n)t}}{\varepsilon - \omega_n}.$$

Under the two qubit product basis,

$$|00\rangle = |0\rangle_1|0\rangle_2, \quad |01\rangle = |0\rangle_1|1\rangle_2, \quad |10\rangle = |1\rangle_1|0\rangle_2, \quad |11\rangle = |1\rangle_1|1\rangle_2,$$

the initially entangled qubits are considered to be in a mixed state with the density matrix [19],

$$\rho_S(0) = \frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \tag{7}$$

where a, b, c are independent parameters governing the nature of the initial state of the two entangled qubits and $z = |z|e^{i\varphi}$ is complex. Note that the entanglement part of the state depends on the initial phase φ and includes the Bell states as special state.

The solution of (6) under the initially state (7) is direct, the density matrix elements of the time evolution are given by [20]

$$\rho_S^{11}(t) = \frac{a}{3}e^{-2\gamma(t)},$$

$$\rho_S^{22}(t) = \frac{a}{3}e^{-\gamma(t)}(1 - e^{-\gamma(t)}) + \frac{b}{6}e^{-\gamma(t)}(1 + e^{-2K^2t^2}) + \frac{c}{6}e^{-\gamma(t)}(1 - e^{-2K^2t^2})$$

$$- \frac{2}{3}k|z|\sin\varphi e^{-2k^2t^2-\gamma(t)}\int_0^t d\tau e^{2k^2\tau^2+\gamma(\tau)},$$

$$\rho_S^{33}(t) = \frac{a}{3}e^{-\gamma(t)}(1-e^{-\gamma(t)}) + \frac{b}{6}e^{-\gamma(t)}(1-e^{-2K^2t^2}) + \frac{c}{6}e^{-\gamma(t)}(1+e^{-2K^2t^2})$$

$$+ \frac{2}{3}k|z|\sin\varphi e^{-2k^2t^2-\gamma(t)}\int_0^t d\tau e^{2k^2\tau^2+\gamma(\tau)},$$

$$\rho_S^{44} = 1 - \frac{a}{3}(e^{-\gamma(t)} - e^{-2\gamma(t)}) - \frac{3-d}{3}e^{-\gamma(t)},$$

$$\rho_S^{23}(t) = \frac{|z|}{3}e^{-\gamma(t)}(\cos\varphi + ie^{-2k^2t^2}\sin\varphi) + i\frac{k(b-c)}{3}e^{-2k^2t^2-\gamma(t)}\int_0^t d\tau e^{2k^2\tau^2+\gamma(\tau)},$$

$$\rho_S^{32}(t) = \rho_S^{23^*}(t).$$

The others are zero. Where

$$\gamma(t) = 2\,\mathrm{Re}\left\{\int_0^t d\tau B(\omega,\tau)\right\}, \tag{8}$$

is decay rate, while $B(\omega, t)$ is related to the spectral density $J(\omega)$.

3 Modulated Entanglement Evolution

In order to analyze how the entanglement evolution is affected by the reservoir, it is necessary to study the measurement of entanglement. From the above density matrix elements, it is easy to calculate the concurrence [21]. We have

$$C(t) = \max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{9}$$

where λ_j are the eigenvalues of the non-Hermitian $W = \rho(t)\bar{\rho}(t)$ arranged in descending order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. Spin-flip operator $\bar{\rho}(t)$ is defined as

$$\bar{\rho}(t) = (\sigma_y^{(1)} \otimes \sigma_y^{(2)})\rho^*(t)(\sigma_y^{(1)} \otimes \sigma_y^{(2)})$$

where the asterisk $\rho^*(t)$ shows that under the normal basis vector $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, it takes the complex conjugate to $\rho(t)$.

For the Lorentzian distribution, the spectral density is expressed as

$$J(\omega) = \frac{\alpha}{2\pi}\frac{\beta^2}{(\omega_1 - \omega)^2 + \beta^2}, \tag{10}$$

where α is a system-environment coupling strength and β is a width of the distribution as an inverse of the reservoir memory time. For $\alpha \gg \beta$, the influence of environment is strongly. This implies that the evolution of system belongs to a non-Markovian process. Under case of $\alpha \ll \beta$, the Markovian approximation may be used to describe the evolution of system. According to the spectral density (8), the decay rate may be expressed as [22]

$$\gamma(t) = \frac{\alpha\beta^2}{(\omega_1-\omega)^2+\beta^2}\left\{t - \frac{1}{\beta}\frac{\beta^2-(\omega_1-\omega)^2}{\beta^2+(\omega_1-\omega)^2}\right\} + \frac{\alpha\beta^2(\omega_1-\omega)^2}{[\beta^2+(\omega_1-\omega)^2]^2}$$
$$\times\left\{\frac{\beta^2-(\omega_1-\omega)^2}{\beta(\omega_1-\omega)^2}\cos(\omega_1-\omega)t - \frac{2}{(\omega_1-\omega)}\sin(\omega_1-\omega)t\right\}e^{-\beta t}. \tag{11}$$

For the initial state in Equation (7) with initial conditions $b = c = |z| = 1$, we get the time dependent concurrence of the two qubits with a Lorentzian bath distribution

$$\tilde{C}(t) = \frac{2}{3}e^{-\gamma(t)}\left[\sqrt{\cos^2\varphi + e^{-4k^4t^4}\sin^2\varphi} - \sqrt{3a - 2(a+a^2)e^{-\gamma(t)} + a^2e^{-2\gamma(t)}}\right]. \tag{12}$$

In the following, we use this formalism to investigate the dynamics of entanglement and modulated entanglement evolution in the initial state (7).

A. $\gamma(t) = 0$

When $\gamma(t) = 0$, i.e., in absence of any environmental perturbation, from (12), we get

$$\tilde{C}(t) = \frac{2}{3}\left[\sqrt{\cos^2\varphi + e^{-4k^4t^4}\sin^2\varphi} - \sqrt{a(1-a)}\right], \tag{13}$$

which is an ideal case of closed quantum systems whose dynamics is only influenced by the initial condition of the entangled qubits. Note that $a = 0.5$, $\sqrt{a(1-a)}$ exists maximum value. So, with an initial phase of 0,

$$\tilde{C}(t) = 2\left[1 - \sqrt{a(1-a)}\right]/3 > 0,$$

in this case there exists no ESD because of absence of the environment. But for another value of the initial phase $\pi/2$, $\tilde{C}(t) = 2[e^{-2k^2t^2} - \sqrt{a(1-a)}]/3$. Because $e^{-k^2t^2}$ is monotonously decreasing function, one must have $\tilde{C}(t) = 0$. In general, if $\cos^2\varphi + e^{-4k^4t^4}\sin^2\varphi \le a(1-a)$, in which case concurrence is zero and the qubits get disentangled. To understand how the interaction would affect the entanglement, we plot the dynamical evolution of entanglement in absence of any environmental perturbation with $\gamma(t) = 0$.

In Fig. 1, combined with above two cases with special phase, we see that the interaction between coupled qubits will reduce the entanglement degree of the system even under the case without external environmental dissipation. With the evolution of the quantum system, the commensal entanglement degree ultimately approaches to a stable value

$$\tilde{C}(t) = \frac{2}{3}\left[|\cos\varphi| - \sqrt{a(1-a)}\right],$$

where, when $|\cos\varphi| < \sqrt{a(1-a)}$, ESD has occurred beyond the stable value. On the other hand, when $|\cos\varphi| > \sqrt{a(1-a)}$, the commensal entanglement degree finally tends to a stable nonzero value.

This may be understood that, for one qubit, another qubit is the external environment. The stronger inter-qubits interaction, the more clearly the decoherence.

B. If $k = 0$

In this case, it is easy to find that, in absence of the inter-qubits interaction, concurrence becomes independent of the initial phase.

$$\tilde{C}(t) = \frac{2}{3}e^{-\gamma(t)}\left[1 - \sqrt{3a - 2(a+a^2)e^{-\gamma(t)} + a^2e^{-2\gamma(t)}}\right]. \tag{14}$$

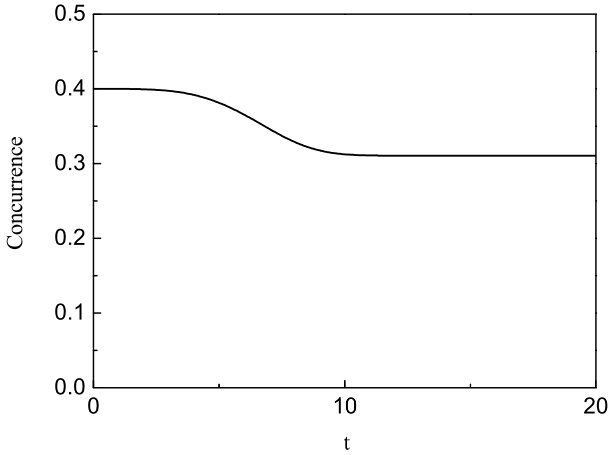


Fig. 1 Concurrence as a function of time with $a=0.2$, $k=0.1$, $\varphi=30°$

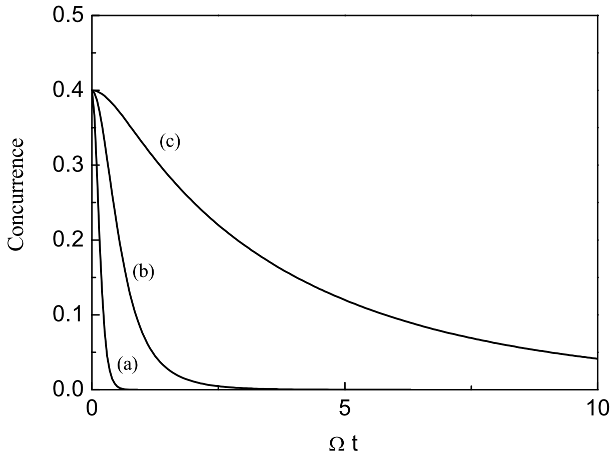


Fig. 2 Concurrence as a function of Ωt for different rates α/β with $a=0.2$. (**a**) $m=10$, (**b**) $m=1$, (**c**) $m=0.1$, where Lorentzian bath distribution is used

Note that $\tilde{C}(t)$ can become negative if

$$3a-2(a+a^2)e^{-\gamma(t)}+a^2e^{-2\gamma(t)}>1,$$

in which case concurrence is zero and the qubits get disentangled.

By setting $\Omega=\omega_1-\omega$, $\beta=2(\omega_1-\omega)$, $\alpha=m\beta$, the concurrence $\tilde{C}(t)$ as a function Ωt is shown at Fig. 2 for different values of the ratio $m=\alpha/\beta$.

In Fig. 2, we show that for $a=0.2$, the non-interacting qubits ($k=0$) exhibit sudden death of entanglement. For the parameters $m=1$ and 10, the entanglement measured by concurrence can disappear during process of the dynamic evolution, i.e., the sudden death of entanglement happens in the thermal reservoir. It is obvious that the entangled degree is dependent on the rate $m=\alpha/\beta$. With increasing of the rate, the ESD become faster. It has pointed out that when $\alpha\gg\beta$, the influence of environment on the system is not only strong but also is a non-Markovian process. According to Fig. 2, therefore, we know that the stronger the non-Markovian process is, the easier the ESD is to occur. Namely, the occurrence of ESD is closely related with the memory effect of the action between environment and quantum system. The curve of $m=0.1$ in Fig. 2 depicts the time evolution of commensal entanglement degree under $\alpha \ll \beta$, which in fact is a Markovian process. The memory effect is indistinct and the ESD occurs gradually. In addition, Fig. 2 indicates that in a given initial state and at thermal equilibrium, the time evolution of the commensal entanglement degree of the system is monotonous and the entanglement recurrence does not occur.

It is known that the interaction always exists in the real coupled qubits and the qubits are in a certain environment. Thus it is easy to know that in general condition ($k \neq 0$, $\gamma(t) \neq 0$), the commensal entanglement degree always decreases and probably occurs ESD during the time evolution. However, ESD may be avoided by manipulating the parameters of the initial entangled state.

C. Bell state

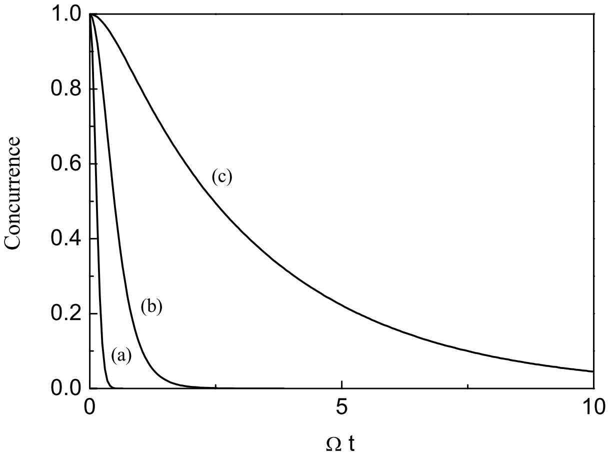
Evidently, the manipulation of entanglement evolution is closely related with the initial state of the quantum state. In the processing of quantum information, an important task is to maintain or improve the entanglement degree of the system so as to avoid the occurrence of ESD. To fulfill this aim, one of strategies may be to select an initial state with maximum entanglement degree after the environment is determined. Therefore the initial state may be chosen as the special Bell states.

Similarly to the parameters in Fig. 2, the concurrence $\tilde{C}(t)$ is presented as a function Ωt in Figs. 3 and 4.

In comparison with Fig. 2, we find that, although the system is initially in Bell state with maximum entanglement degree, the overall evolving trend of commensal entanglement degree is similar to that in Fig. 2 under the effect of environmental dissipation. The maximum entanglement degree occurs only at the initial time. With time evolution, the entanglement degree decrease monotonously, especially for non-Markovian process where ESD is easier to occur.

It is necessary to emphasize that when the system is initially in $(|00\rangle + |11\rangle)/\sqrt{2}$, $\tilde{C}(t) = e^{-2\gamma(t)}$; while when the system is initially in $(|01\rangle + |10\rangle)/\sqrt{2}$, then $\tilde{C}(t) = e^{-\gamma(t)}$. Obviously, different initial Bell state corresponds to different time evolution of the commensal entanglement degree. But an extremely important similarity is that the commensal entanglement degree has no relation with the interaction between the qubits.

Fig. 3 Concurrence as a function of Ωt for different rates α/β with $a = 0.2$. (**a**) $m = 10$, (**b**) $m = 1$, (**c**) $m = 0.1$. The initial state is $(|00\rangle + |11\rangle)/\sqrt{2}$



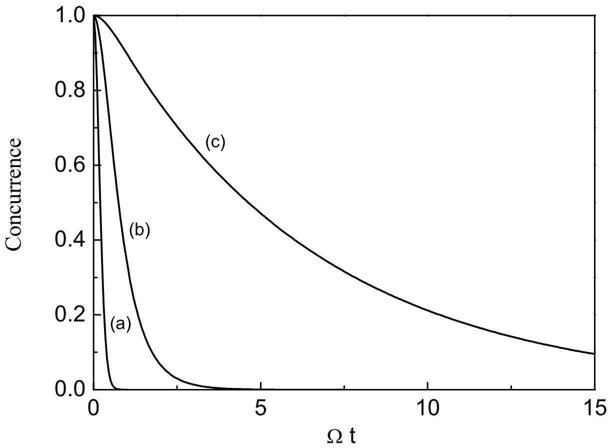


Fig. 4 Concurrence as a function of Ωt for different rates α/β with $a = 0.2$. (**a**) $m = 10$, (**b**) $m = 1$, (**c**) $m = 0.1$. The initial state is $(|01\rangle + |10\rangle)/\sqrt{2}$

4 Conclusions

In summary, the influence of non-Markovian environment on the entanglement decoherence of coupled superconducting qubits is investigated in detail. The commensal entanglement degree of Markovian process and non-Markovian process is also compared. The results indicate that:

(1) The entanglement degree of quantum system not only relies on the initial state of the quantum system, but also strongly relies on the environment. Owing to the memory feedback effect of the environment action, the entanglement of the coupled qubits at the thermal equilibrium always tends suddenly to zero differently from the process of Markovian process.

(2) Generally speaking, the interaction between qubits will lead to the reduction of the entanglement degree, which depends on the initial state (for example, Bell state) or particular initial phase angle. But the influence of the interaction between qubits on the system entanglement degree can be avoided.

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